EXERCISE 1.1

1. If $a, b \in \mathbf{R}$ and $a + b = 0$, prove that $a = -b$.
2. Prove that $(-a)(-b) = ab$ for all $a, b \in \mathbf{R}$.
3. Prove that $\big|\,|a| - |b|\,\big| \leq |a - b|$ for every $a, b \in \mathbf{R}$.
4. √Express $3 < x < 7$ in modulus notation.
5. Let $\delta > 0$ and $a \in \mathbf{R}$. Show that $a - \delta < x < a + \delta$ if and only if $|x - a| < \delta$.
6. Give an example of a set of rational numbers which is bounded above but does not have a rational Sup.

Solve each of the following (**Problems 7 – 15**):

7. √ $|2x + 5| > |2 - 5x|$
8. $\left|\dfrac{x+8}{12}\right| < \dfrac{x-1}{10}$
9. √ $|x| + |x - 1| > 1$
10. $12x^2 - 25x + 12 > 0$
11. √ $\dfrac{x-1}{2} - \dfrac{1}{x} > \dfrac{4}{x} + 5$
12. $|x^2 - x + 1| > 1$
13. √ $x^{-2} - 4x^{-1} + 4 > 0$
14. $\dfrac{2x}{x+2} \geq \dfrac{x}{x-2}$
15. √ $x^4 - 5x^3 - 4x^2 + 20x \leq 0$.
16. The cost function $C(x)$ and the revenue function $R(x)$ for producing x units of a certain product are given by
$$C(x) = 5x + 350, \quad R(x) = 50 - x^2.$$
Find the values of x that yield a profit.

Function from $\mathbf{R}$ to $\mathbf{R}$ is defined by the given formula. Determine the domain of the function (**Problems 17 – 22**)

17. $f(x) = \sqrt{1 - x^2}$
18. $f(x) = \dfrac{a+x}{a-x}$
19. $f(x) = \dfrac{1}{\sqrt{(1-x)(2-x)}}$
20. $f(x) = \sqrt{3+x} + \sqrt{7-x}$
21. $f(x) = \begin{cases} x^2 - 1 & \text{if } x \leq 2 \\ \sqrt{x-1} & \text{if } x > 2 \end{cases}$ and find $f(2)$.
22. $f(x) = \sqrt{\dfrac{x-4}{x+1}}$

Draw the graphs of the following functions (**Problems 23 – 30**):

23. $f(x) = [x] + [x-1]$, for all $x \in \mathbf{R}$

24. $f(x) = [x] + [x+1]$, for all $x \in \mathbf{R}$

25. $f(x) = x - [x]$, for $x \in [-3, 3]$ **[Saw Tooth Function]**

26. $f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x < 0 \\ -\dfrac{1}{x} & \text{if } x > 0 \end{cases}$

27. $f(x) = x^2 + 2x - 1$, for all $x \in \mathbf{R}$.

28. $f(x) = \dfrac{1}{x^2}$, $x \neq 0$

29. $f(x) = \dfrac{1}{x}$, $x \neq 0$

30. $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$

This is known as **signum (sgn)** function.

Find the Sup and Inf (if they exist) of the given set (**Problems 31 – 34**):

31. $\left\{ (-1)^n \left(1 - \dfrac{1}{n}\right), n = 1, 2, 3 \ldots \right\}$

32. The set of all nonnegative integers.

33. The set $A = \{x \in \mathbf{R} : 0 < x \leq 3\}$

34. The set $B = \{x \in \mathbf{R} : x^2 - 2x - 3 < 0\}$

Sketch the graph of the given equation. Also determine which is the graph of a function (**Problems 35 – 38**):

35. $y^2 = x$

36. $|x| = |y|$

37. $x^2 + y^2 = 9$

38. $y = |x| + x$

39. Find formulas for the functions $f + g, fg$ and $\dfrac{f}{g}$, where

$$f(x) = \sqrt{x^2 - 1}, \quad g(x) = \frac{1}{\sqrt{4 - x^2}}$$

Also write the domain of each of these functions.

40. Find formulas for $f \circ g$ and $g \circ f$, where

$$f(x) = \sqrt{x^3 - 3}, \quad g(x) = x^2 + 3.$$

Exercise 1.1

1:- If $a, b \in R$ and $a+b=0$, Prove that $a=-b$

Sol: Since $b \in R$ (given)

So there exist $-b \in R$

s.t $b+(-b)=0 \rightarrow (1)$

$\because$ $a+b=0$ (given)

Adding $(-b)$ both Sides

$a+b+(-b) = 0+(-b)$

$a+(b+(-b)) = -b$ (by Associative Law

$a+0 = -b$ (by Add. Inverse Law

$a = -b$ (by + Identity Law

2:- Prove that $(-a)(-b)=ab \quad \forall a,b \in R$

Sol:-

$(-a)(-b)-ab = (-a)(-b)+(-ab)$ (Def: of Subt:)

$= (-a)(-b)+(-a)b$ $\quad \because (-a)b=-ab$

$= (-a)(-b+b)$ (Left Dist: Law)

$= (-a)(0)$ (Inverse Law)

$= 0$ $\quad \because a \cdot 0 = 0$

So $(-a)(-b)-ab=0$

$\Rightarrow (-a)(-b)=ab$ $\quad \because x-y=0 \Rightarrow x=y$

3:- Prove $\big||a|-|b|\big| \leq |a-b| \quad \forall a,b \in R$

Sol: Here

$|a| = |a-b+b|$ $\quad$ +& $-b$

$\leq |a-b|+|b|$

$|a| \leq |a-b|+|b|$

$|a|-|b| \leq |a-b| \rightarrow (1)$

Again $|b| = |b-a+a|$ $\quad$ +&- by a

$\leq |b-a|+|a|$

$|b|-|a| \leq |b-a|$

$|b|-|a| \leq |a-b|$

$\times$ by -1 on both Sides

$|a|-|b| \geq -|a-b| \rightarrow (2)$

From (1) & (2)

$-|a-b| \leq |a|-|b| \leq |a-b|$

$\Rightarrow \big||a|-|b|\big| \leq |a-b|$ Prove

$\because |x| \leq a$

$\Rightarrow -a \leq x \leq a$

4 Express $3<x<7$ in modulus notation

Sol: We know

$$|x-a|<b$$

$$\Rightarrow -b<x-a<b$$

$$\Rightarrow a-b<x<a+b \rightarrow (1) \quad \text{Adding } 'a',$$

Also given $3<x<7 \rightarrow (2)$

Comparing (1) & (2)

$$a-b=3 \quad \& \quad a+b=7$$

Adding

$$\begin{array}{r} a-\cancel{b}=3 \\ a+\cancel{b}=7 \\ \hline 2a=10 \\ \boxed{a=5} \end{array}$$

Sub

$$\begin{array}{r} \cancel{a}-b=3 \\ -\cancel{a}\underset{-}{+}b=\underset{-}{7} \\ \hline -2b=-4 \\ b=\frac{-4}{-2} \end{array} \quad \boxed{b=2}$$

So Required Mod Notation is

$$|x-a|<b \Rightarrow |x-5|<2$$

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5 Let $\delta>0$ and $a\in R$ Show that $a-\delta<x<a+\delta$ iff $|x-a|<\delta$

Sol: Let $a-\delta<x<a+\delta$

$$a-\delta-a<x-a<a+\delta-a \quad \because \text{Sub } 'a',$$

$$-\delta<x-a<\delta$$

$$\Rightarrow |x-a|<\delta \quad \text{By def: of Mod.}$$

Conversely let $|x-a|<\delta$

$$\Rightarrow -\delta<x-a<\delta \quad \text{By def: of mod.}$$

$$\Rightarrow -\delta+a<x-a+a<\delta+a \quad \text{Adding } 'a',$$

$$\Rightarrow -\delta+a<x<\delta+a$$

$$\Rightarrow a-\delta<x<a+\delta$$

Proved.

So $a-\delta<x<a+\delta$ iff $|x-a|<\delta$.

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6) Give an example of a Set of Rational numbers which is bounded above but does not have a Rational Supremum

Sol: Consider a Set A of Rational number defined by

$$A = \{ x \in \mathbb{Q} : x^2 < 2 \}$$

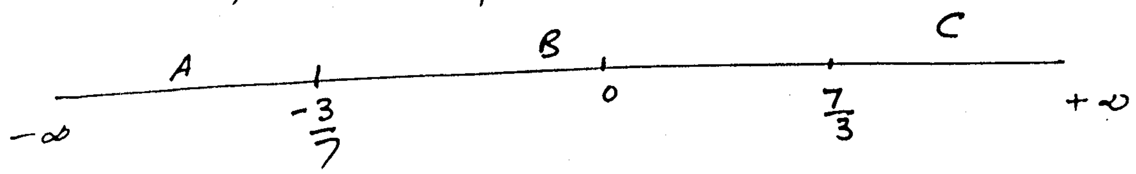
It is obvious that Set A is bounded above but it does not have Rational Sup. Because its Sup is $\sqrt{2}$ which is Irrational.

Q7 Solve $|2x+5| > |2-5x| \rightarrow (1)$

Sol: Associate eq.

$$2x+5 = \pm (2-5x)$$

$2x+5 = 2-5x$	$2x+5 = -(2-5x)$
$2x+5x = 2-5$	$2x+5 = -2+5x$
$7x = -3$	$7 = 3x$
$x = \frac{-3}{7}$	$x = \frac{7}{3}$



For Region A Put $x = -1$ in (1) $|-2+5| > |2+5|$ False

For Region B Put $x = 1$ in (1) $|2+5| > |2-5|$ True

For Region C Put $x = 3$ in (1) $|6+5| > |2-15|$ False

Hence Solution Set is $\{ x : \frac{-3}{7} < x < \frac{7}{3} \} =]-\frac{3}{7}, \frac{7}{3}[$

Q8 $\left| \frac{x+8}{12} \right| < \frac{x-1}{10} \rightarrow (1)$

Associate Eq $\frac{x+8}{12} = \pm \left(\frac{x-1}{10} \right)$

$$\frac{x+8}{12} = \frac{x-1}{10}$$

$$10x + 80 = 12x - 12$$

$$\Rightarrow 92 = 2x$$

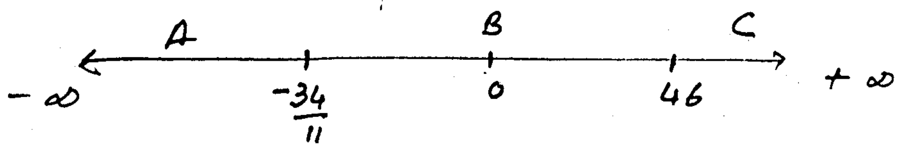
$$\Rightarrow \boxed{x = 46}$$

$$\frac{x+8}{12} = -\frac{(x-1)}{10}$$

$$10x + 80 = -12x + 12$$

$$22x = -68$$

$$x = \frac{-68}{22} = -\frac{34}{11} = -3.09$$



Region A, Put $x = -4$ in (1) $\left|\frac{-4+8}{12}\right| < -\frac{5}{10}$

$\frac{1}{3} < -\frac{1}{2}$ (False)

Region B Put $x=0$ in (1) $\left|\frac{8}{12}\right| < -\frac{1}{10}$

$\frac{2}{3} < -\frac{1}{10}$ (False)

Region C Put $x = 50$ in (1) $\left|\frac{58}{12}\right| < \frac{49}{10}$

$4.83 < 4.9$ True.

Hence $S.S =]46, \infty[= \{x \mid x > 46\}$

(9) $|x| + |x-1| > 1$

Associate eq. $|x| + |x-1| = 1$

$\pm x \pm (x-1) = 1$

(+ +)

$+x + (x-1) = 1$

$2x - 1 = 1$

$2x = 2$

$\boxed{x = 1}$

(- , -)

$-x - (x-1) = 1$

$-x - x + 1 = 1$

$-2x = 0$

$\boxed{x = 0}$

(+ -)

$x - (x-1) = 1$

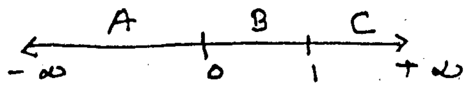
$1 = 1$

(- +)

$-x + x - 1 = 1$

$-1 = 1$

Impossible.



Region A Put $x=-1$ in (1) $1+2 > 1$ (True)

Region B Put $x = \frac{1}{2}$ in (1) $\left|\frac{1}{2}\right| + \left|-\frac{1}{2}\right| > 1$

$1 > 1$ (False)

Region C Put $x=2$ in (1) $|2| + |2-1| > 1$

$2 + 1 > 1$ (True)

$S.S = \,]-\infty, 0[\; \cup \;]1, \infty[$ — Ans

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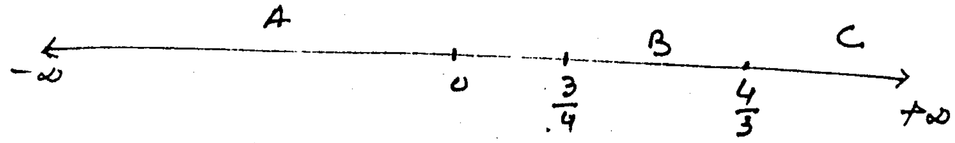
(10) $12x^2 - 25x + 12 > 0 \rightarrow (1)$

Associate Eq. of (1) is

$12x^2 - 25x + 12 = 0$

$x = \frac{25 \pm \sqrt{625-576}}{24} = \frac{25 \pm 7}{24} = \frac{4}{3}, \frac{3}{4}$ are boundary Number for (1)

The number line will be divided into the region as show in fig



Region A test $x = 0$ in (1) $12 > 0$ (True)

Region B test $x = 1$ in (1) $-1 > 0$ (False)

Region C test $x = 2$ in (1) $48 - 50 + 12 > 0$ (True)

So the Solution Set is

$\{x : x < \frac{3}{4}\} \cup \{x : x > \frac{4}{3}\} = \,]-\infty, \frac{3}{4}[\; \cup \;[\frac{4}{3}, \infty[$

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(11) $\frac{x-1}{2} - \frac{1}{x} > \frac{4}{x} + 5$

Associate Eq of (1) is

$\frac{x-1}{2} - \frac{1}{x} = \frac{4}{x} + 5$

or $\frac{x^2 - x - 2}{2x} = \frac{4 + 5x}{x}$

by x multiply.

$\cancel{x}(x^2 - x - 2) = 2\cancel{x}(4 + 5x)$

$x^3 - x^2 - 2x = 8x + 10x^2$

$\Rightarrow x^3 - 11x^2 - 10x = 0$

$$\Rightarrow x(x^2 - 11x - 10) = 0$$

$$\Rightarrow x = 0 \text{ and } x^2 - 11x - 10 = 0$$

$$x = \frac{11 \pm \sqrt{121 + 40}}{2}$$

$$= \frac{11 \pm \sqrt{161}}{2} = \frac{11 \pm 12.68}{2}$$

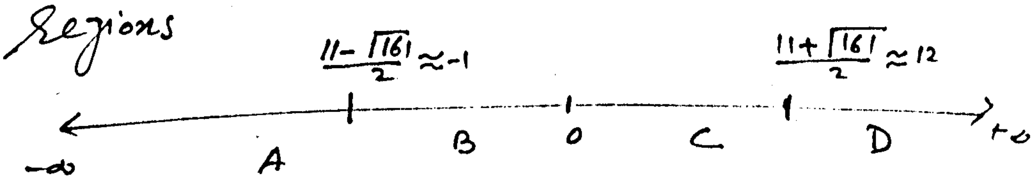
$$= \frac{23.68}{2}, \quad \frac{-1.68}{2}$$

$$= 11.84, \quad -.84 \quad \text{(are only Boun. Number)}$$

$$\approx 12, \quad -1$$

Note
0 is free boundary number because at $x = 0$ the denominator of (1) vanishes
i.e $\frac{x^2 - x - 2}{0} = \frac{4 + 5x}{0}$
So '0' can not be Sol: Set

So the number line is divided into distinct Regions



Region A test $x = -2$ in (1)

$$\frac{-2-1}{2} + \frac{1}{2} > -\frac{4}{2} + 5$$

$$\text{or} \quad -\frac{3}{2} + \frac{1}{2} > 3 \quad \text{(False)}$$

Region B test $x = -\frac{1}{2}$ in (1)

$$\frac{-\frac{1}{2} - 1}{2} - \frac{1}{-\frac{1}{2}} > \frac{4}{-\frac{1}{2}} + 5$$

$$-\frac{3}{4} + 2 > -8 + 5$$

$$\frac{5}{4} > -3 \quad \text{(True)}$$

Region C test $x = 10$ in (1)

$$\frac{10-1}{2} - \frac{1}{10} > \frac{4}{10} + 5$$

$$\frac{9}{2} - \frac{1}{10} > \frac{54}{10}$$

$$\frac{44}{10} > \frac{54}{10} \quad \text{(False)}$$

Region D test $x = 15$ in (1)

$$\frac{15-1}{2} - \frac{1}{15} > \frac{4}{15} + 5$$

$$7 - \frac{1}{15} > \frac{4}{15} + 5$$

$$\frac{104}{15} > \frac{79}{15} \quad \text{(True)}$$

We See that Region B & D are Solution Set

So Solution Set is $\left]\frac{11-\sqrt{161}}{2}, 0\right[\cup \left]\frac{11+\sqrt{161}}{2}, \infty\right[$

(12) $|x^2-x+1| > 1 \longrightarrow (1)$

Associated Eq of (1) is

$|x^2-x+1| = 0$

$x^2-x+1 = \pm 1$

$x^2-x+1 = 1$

$x^2-x = 1-1$

$x^2-x = 0$

$x(x-1) = 0$

$x = 0, 1$

$x^2-x+1 = -1$

$x^2-x-2 = 0$

$x = \frac{1 \pm \sqrt{1-8}}{2}$

$= \frac{1 \pm \sqrt{7}\, i}{2}$

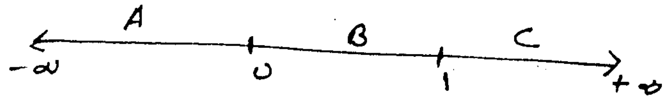
Since both $\frac{1+i\sqrt{7}}{2}$ & $\frac{1-i\sqrt{7}}{2}$ are Complex number

And Can not Represented by a number line.

Thus They are not boundary numbers.

There are only two boundary number "0" 1

So the number line is divided into Regions



Region A test $x=-1$ in (1) $|1+1+1| > 1$ (True.)

Region B test $x=\frac{1}{2}$ in (1) $|\frac{1}{4}-\frac{1}{2}+1| > 1$

$|\frac{3}{4}| > 1$ (False)

Region C test $x=2$ in (1) $|4-2+1| > 1$ (True)

S.S is $]-\infty, 0[\cup]1, \infty[$

(13) $x^{-2} - 4x^{-1} + 4 > 0 \rightarrow (1)$ or $\frac{1}{x^2} - \frac{4}{x} + 4 > 0 \rightarrow (1)$

Associated Eq of (1) is

$\frac{1}{x^2} - \frac{4}{x} + 4 = 0 \Rightarrow \frac{1-4x+4x^2}{x^2} = 0$

$\Rightarrow 1 - 4x + 4x^2 = 0$

$\Rightarrow 4x^2 - 4x + 1 = 0$

$\Rightarrow (2x-1)^2 = 0$

$\Rightarrow x = 1/2$ is B. Number.

Best

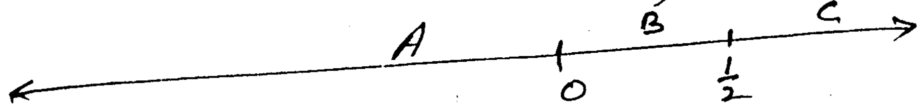
$\because (1)\ \frac{1}{x^2} - \frac{4}{x} + 4 > 0$

$\Rightarrow \left(\frac{1-2x}{x}\right)^2 > 0 \rightarrow (1)$

for Inequality given in (1)

Now denominator of $\frac{1-4x+4x^2}{x^2}$ is zero at $x=0$

So $x=0$ free boundary number



Region A test $x=-1$ $\left(\frac{1+2}{-1}\right)^2 > 0$ True

Region B " $x = \frac{1}{4}$ $\left[\frac{1-\frac{1}{2}}{1/4}\right]^2 > 0$ True.

Region C " $x = 1$ $\left(\frac{1-2}{1}\right)^2 > 0$ True.

The Solution Set is

$\{x : x < 0\} \cup \{x : 0 < x < \frac{1}{2}\} \cup \{x : x > \frac{1}{2}\}$

$=]-\infty, 0[\ \cup\ ]0, \frac{1}{2}[\ \cup\ ]\frac{1}{2}, \infty[$

(14) $\frac{2x}{x+2} \geq \frac{x}{x-2} \rightarrow (1)$

Sol: $x = -2, 2$ are free boundary number

The associated eq of (1) will be,

$$\frac{2x}{x+2} = \frac{x}{x-2}$$

$\Rightarrow 2x(x-2) = x(x+2)$

$\Rightarrow 2x^2 - 4x = x^2 + 2x$

$\Rightarrow x^2 - 6x = 0$

$\Rightarrow x(x-6) = 0$

$\Rightarrow x = 0, 6$ are the boundary numbers for (1)

The boundary numbers divide the number line into Regions as shown.

A -2 B 0 C 2 D 6

Region A, test $x = -3$ in (1) $\frac{-6}{-3+2} \geq \frac{-3}{-3-2}$

$6 > \frac{3}{5}$ (True)

Region B test $x = -1$ in (1) $\frac{-2}{-1+2} \geq \frac{-1}{-1-2}$

$-2 \geq \frac{1}{3}$ (False)

Region C test $x = 1$ in (1) $\frac{2}{3} \geq \frac{1}{1-2}$ (True)

Region D test $x = 3$ in (1) $\frac{6}{5} \geq \frac{3}{1}$ False)

Region E test $x = 7$ in (1) $\frac{14}{9} \geq \frac{7}{5}$

$1.55 \geq 1.4$ (True)

Solution Set is Union.

$]-\infty, -2[\ \cup \ [0, 2[\ \cup \ [6, \infty[$ Ans

OR $\{x : x < -2\} \cup \{x : 0 \leq x < 2\} \cup \{x : x \geq 6\}$ Ans

Q15 $x^4 - 5x^3 - 4x^2 + 20x \leq 0 \longrightarrow (1)$

Sol: Associated Eq of (1) is

$x^4 - 5x^3 - 4x^2 + 20x = 0$

$x(x^3 - 5x^2 - 4x + 20) = 0$

$x(x^2(x-5) - 4(x-5)) = 0$

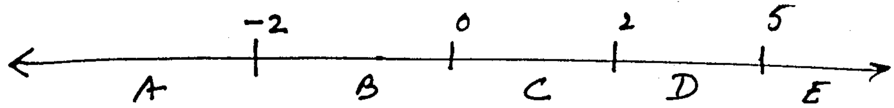
$x(x^2-4)(x-5) = 0$

$x(x-2)(x+2)(x-5) = 0$

$x = 0, 2, -2, 5$

are the Boundary number for (1)

Locate the boundary numbers on a numbers line and check each region whether it belongs to the solution set or not.



Region A, test $x=-3$ in ① $81+135+36-60 \le 0$ (False)

Region B, test $x=-1$ in ① $1+5-4-20 \le 0$ (True)

Region C, test $x=1$ in ① $1-5-4+20 \le 0$ (False)

Region D, test $x=3$ in ① $81-135-36+60 \le 0$ (True)

Region E, test $x=6$ in ① $1296-1080-144+120 \le 0$ True

Sol: Set is $\{x: -2 \le x \le 0\} \cup \{x: 2 \le x \le 5\}$

$$= [-2, 0] \cup [2, 5]$$

⑯ The Cost function $C(x)$ and the Revenue function $R(x)$ for producing x units of certain product are given

$C(x) = 5x + 350$

$R(x) = 50 - x^2$

i, Find the values of x that yields a Profit.

Extra ii, Find the values of x that results in a Loss.

Solution A Profit is produced if Revenue exceeds Cost

For Profit

$$\text{Revenue} > \text{Cost}$$

$$R(x) > C(x)$$

$$50x - x^2 > 5x + 350$$

$$0 > x^2 - 50x + 5x + 350$$

$$0 > x^2 - 45x + 350$$

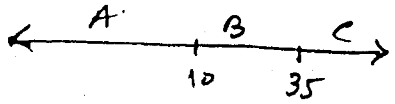
$$\Rightarrow x^2 - 45x + 350 < 0 \longrightarrow ①$$

Associated Eq: $x^2 - 45x + 350 = 0$

$$x^2 - 35x - 10x + 350 = 0$$

$$(x-10)(x-35)=0$$

$$x = 10, \ 35 \ (B.N)$$



<u>for Region A</u> Put $x=0$ in ① $0 > 350$ (False)

<u>for Region B</u> Put $x=15$ in ①

$$0 > 15^2 - 45(15) + 350$$

$$0 > 225 - 675 + 350$$

$$0 > -100 \quad \text{(True)}$$

<u>for Region C</u> Put $x=40$ in ①

$$0 > 40^2 - 45(40) + 350$$

$$0 > 1600 - 1800 + 350$$

$$0 > 150 \quad \text{(False)}$$

Thus the values of x that gives a Profit are

$$\{x : 10 < x < 35\}$$

ii Extra <u>For Loss</u>

Cost > Revenue

$$C(x) > R(x)$$

$$5x + 350 > 50x - x^2$$

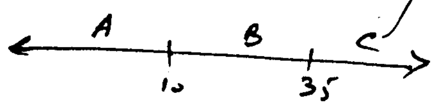
$$\Rightarrow x^2 - 45x + 350 > 0 \rightarrow ①$$

ASS: Eq ①

$$x^2 - 45x + 350 = 0$$

$$(x-10)(x-35) = 0$$

$x = 10, 35$ Boundary No:



<u>Regi A</u> Put $x=0$ in ① $350 > 0$ (True)

<u>Regi B</u> Put $x=15$ in ①

$$-100 \ngtr 0 \quad \text{(False)}$$

<u>Region C</u> Put $x=40$ in ①

$$150 > 0 \quad \text{True.}$$

Hence the values of x that results in Loss are

$$\{x : x < 10\} \cup \{x : x > 35\}$$

Where x is the Integer.

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(17) Function f from R to R is defined by the given formula. Determine the domain of the function.

(17) $f(x) = \sqrt{1-x^2}$

$f(x)$ will be Real if

$$1 - x^2 \geq 0$$

$$-x^2 \geq -1$$

$$x^2 \leq +1$$

$$\Rightarrow \pm x \leq 1$$

$$x \leq 1 \quad \& \quad -x \leq 1$$

$$x \leq 1 \qquad x \geq -1$$

$$-1 \leq x \leq 1$$

$$\Rightarrow |x| \leq 1$$

when $|x| > 1$ $f(x)$ will be Complex $\Rightarrow$ for $|x| \leq 1$ has Real Values

Hence dom of f is $|x| \leq 1$

(18) $f(x) = \frac{a+x}{a-x}$

Sol: $f(x)$ will be infinite when $x=a$

Dom. of $f = \mathbb{R} - (a)$

or Set of all real number except $x = a$

(19) $f(x) = \frac{1}{\sqrt{(1-x)(2-x)}}$

Sol we see that when we put $x=1,2$

$f(x)$ will be undefined.

So domain of f is Set of real number except $x \in [1, 2]$

dom $f = R - [1, 2]$

$\because 1 \le x \le 2$ $f(x)$ become imaginary

(20) $f(x) = \sqrt{3+x} + \sqrt{7-x}$ —(1)

Sol $f(x)$ will be real if

$7 - x \geq 0$

$\Rightarrow 7 \geq x$

$\Rightarrow x \leq 7$

& $3+x \geq 0$

$x \geq -3$

$\Rightarrow$ when $x > 7$ (1) become Imaginary

also when $x < -3$ (1) become Imaginary

So domain of f is Set of real number (x) Such that

$x \leq 7$ & $x \geq -3$

$\therefore x \in [-3, 7]$

Q(21)

$$f(x) = \begin{cases} x^2-1 & \text{if } x \leq 2 \\ \sqrt{x-1} & \text{if } x > 2 \end{cases}$$

Sol We see that the given function is defined for all real values of x

So domain of f is R.

Extra $f(2) = (2)^2 - 1 = 4 - 1 = 3$

Q(22) $f(x) = \sqrt{\frac{x-4}{x+1}}$

Sol. We see that f is not defined at $x = -1$

Also if $-1 < x < 4$ Then again $f(x)$ becomes imaginary

Hence domain of $f(x)$ is Set of all real numbers except when $x \in [-1, 4[$ $= -1 \le x < 4$

i.e $R - [-1, 4[$

(23) Draw the graphs of the following fn:-

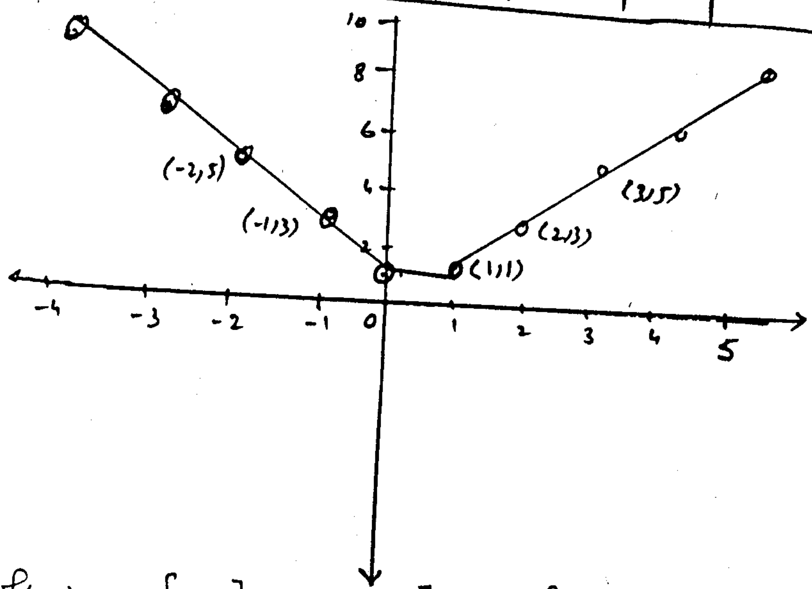
<u>Note</u> graph is Function When vertical line cut the graph at One pt:

$f(x) = |x| + |x-1|$ for all $x \in R$

$$= \begin{cases} x + x - 1 = 2x-1 & \text{when } x \geq 0 \\ -x - x + 1 = -2x+1 & \text{when } x < 0 \end{cases}$$

Some Table values of given function are

x	0	-1	-2	-3	1	2	3	-4	4	5
$y=f(x)$	1	3	5	7	1	3	5	9	7	9



(24) $f(x) = [x] + [x+1]$ for all $x \in R$

Note (1) Here $[x]$ denotes greatest integer or Bracket function not greater than x. Since x is an integer so values of $f(x)$ are also integers. Now if n is an integer and $n \leq x < n+1$ then $[x] = n$ and so f is constant on $[n, n+1]$

Note (2) The right hand end pts of segments of lines are not part of the graph.

Hence from $f(x) = [x] + [x+1]$

$y = f(x) = 1$ when $0 \le x < 1$

$= 3$ $\quad 1 \le x < 2$

$= 5$ $\quad 2 \le x < 3$

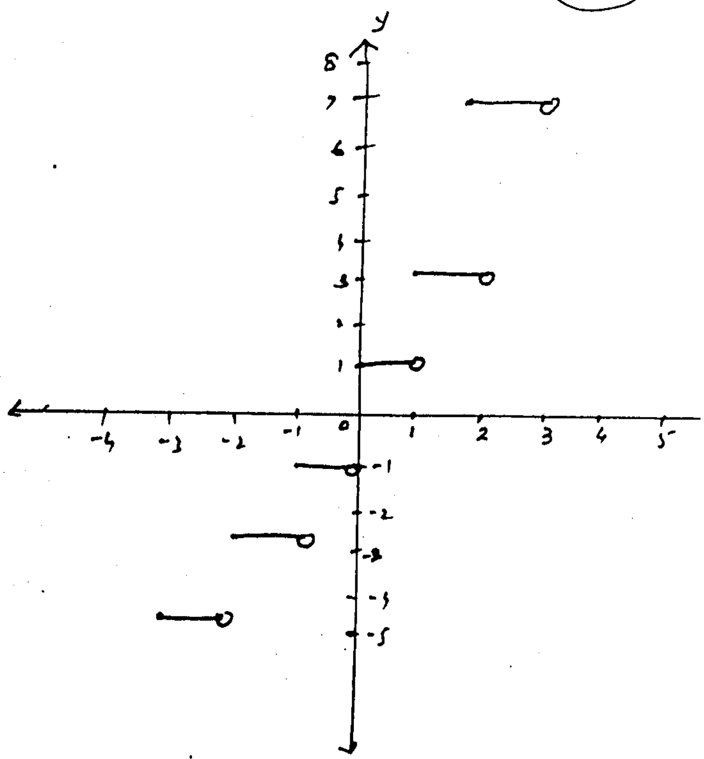
$= 9$ $\quad 3 \le x < 4$

$= 9$ $\quad 4 \le x < 5$

$= -1$ $\quad -1 \le x < 0$

$= -3$ $\quad -2 \le x < -1$

$= -5$ $\quad -3 \le x < -2$



Note $f(x) = [x] + [x+1]$

$= [0] + [0+1] = 1, 0 \le x < 1 \Rightarrow (0,1)\ (.1,1)\ (.2,1) \dots (.9,1)$

$= [1] + [1+1] = 3, 1 \le x < 2 \Rightarrow (1,3)\ (1.1,3)\ (1.2,3) \dots (1.9,3)$

$= [-1] + [-1+1] = -1, -1 \le x < 0 \Rightarrow (-1,-1), (-.9,-1)\ (-.8,-1) \dots (-.1,-1)$

(25) $f(x) = x - [x]$ for all $x \in [-3, 3]$

(Saw — tooth function)

Sol. When x is an integer (whether +ve or -ve)

Then $f(x) = 0$ e.g $x = \pm 3, \pm 2, \pm 1, 0$

When $x = -3$ $\quad f(x) = -3 - [-3] = -3 + 3 = 0$

When $x = 3$ $\quad f(x) = 3 - [3] = 3 - 3 = 0$

When $x = 2$ $\quad f(x) = 2 - [2] = 2 - 2 = 0$

Similarly for other integral values of $x \in [-3, 3]$ $f(x) = 0$

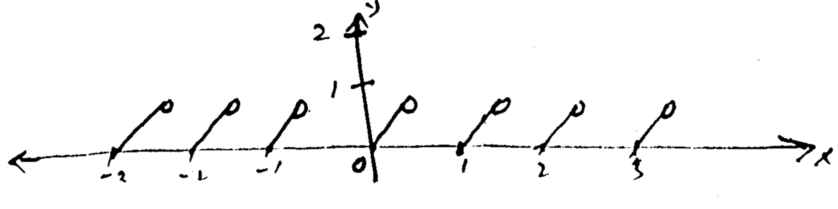
<u>When x is not integer</u>

Let $x = 2.5$ $\quad f(x) = 2.5 - [2.5] = 2.5 - 2 = .5$

When $x = -2.5$ $\quad f(x) = -2.5 - [-2.5] = -2.5 - (-3) = -2.5 + 3 = .5$

When $x = 1.5$ $\quad f(x) = 1.5 - [1.5] = 1.5 - 1 = .5$

When $x = 1.5$ $\quad f(x) = -1.5 - [-1.5] = -1.5 - (-2) = -1.5 + 2 = .5$



x	0	± 1	1.5	-1.5	± 2	± 2.5
$f(x)$	0	0	.5	.5	0	.5

Note $[-n.n_1n_2] = -n-1$; $[n.n_1.n_2] = n$ By definition of Bracket fn

(26) $f(x) = \frac{1}{x}$ if $x < 0$

$= -\frac{1}{x}$ if $x > 0$

Sol: We see that at $x=0$ $f(0)$ is undefined. i.e ∞

x is -ive $f(x)$ is -ive and when x is +ve $f(x)$ is also -ve.

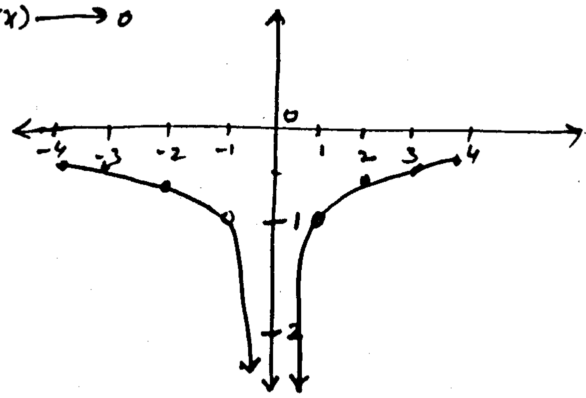
and value of $f(x)$ increases as x decreases.

Value of $f(x)$ decreases as x increases.

When $x \to 0$, then $f(x) \to -\infty$ both sides of y-axis

when x is very large, then $f(x) \to 0$

x	0	1	-1	2	-2	3	-3	4	-4
y	$-\infty$	-1	-1	-.5	-.5	-.33	-.33	-.25	-.25



Q(27) $f(x) = x^2+2x-1 \quad \forall\ x \in R$ → ①

Sol: ① Can be written as

$y = x^2+2x-1 = x^2+2x+1-2$

$y = (x+1)^2-2$

$\Rightarrow y+2 = (x+1)^2$ → ②

Put $x+1 = X'$, $y+2 = y'$

So ② will be

$Y' = X'^2$ → ③

Eq ③ represents a parabola symmetric about y-axis (Eq ③ remains same when we put $x=-x$)

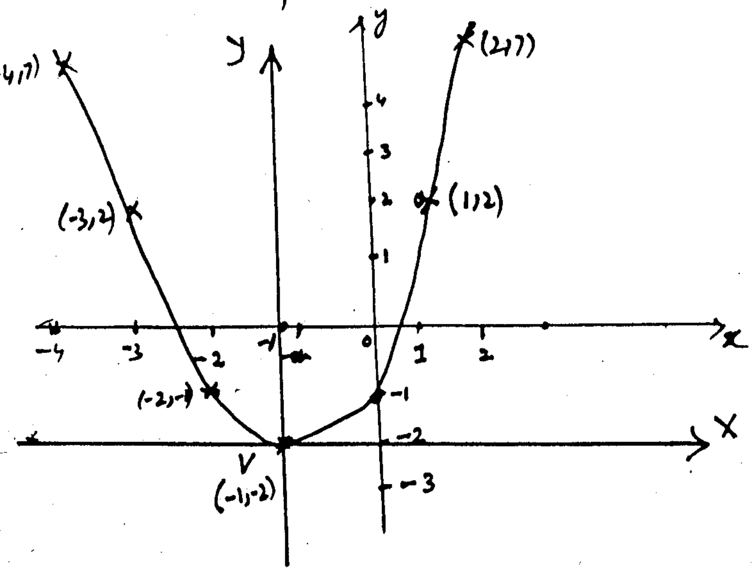
x	0	1	-1	2	-2	-3	-4
f(x)	-1	2	-2	7	-1	2	7

Vertex $X'=0$ $Y'=0$

$x+1=0$ $y+2=0$

$x=-1$ $y=-2$

$V(-1,-2)$



(28) $f(x) = \frac{1}{x^2} \rightarrow (1) \quad x \neq 0$

Sol → Eq (1) Can be written as

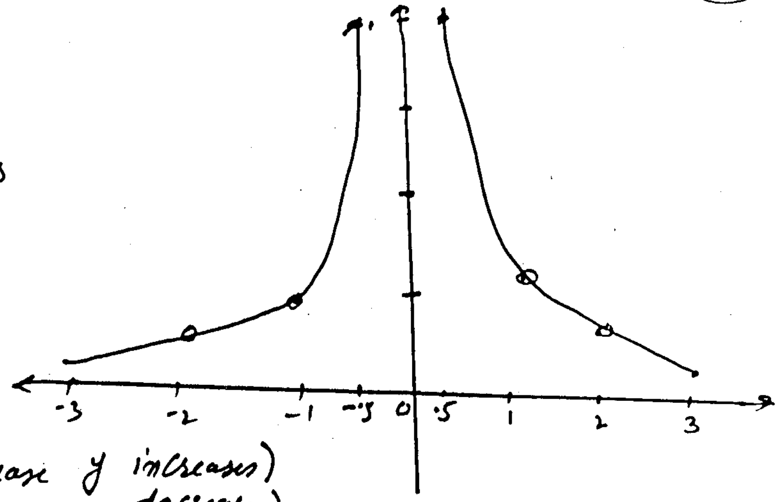
$y = \frac{1}{x^2} \rightarrow (2)$

Eq (2) gives that y is +ve for all values of x.

y is alway +ve graph lies entirly above x-axis (x decrease y increases, x increase y decreases)

at $x = 0$ $f(x) = \infty$

Put $x = -x$ in Eq (2) eq: does not change. Implies that graph is symmetric about y-axis i.e lies on both side of y-axis



x	1	-1	2	-2	3	±.5
y	1	1	.25	.25	.11	4

(29) $f(x) = \frac{1}{x} \quad x \neq 0$

We see that function is defined at all value of x except $x = 0$

When x is +ve y is also +ve

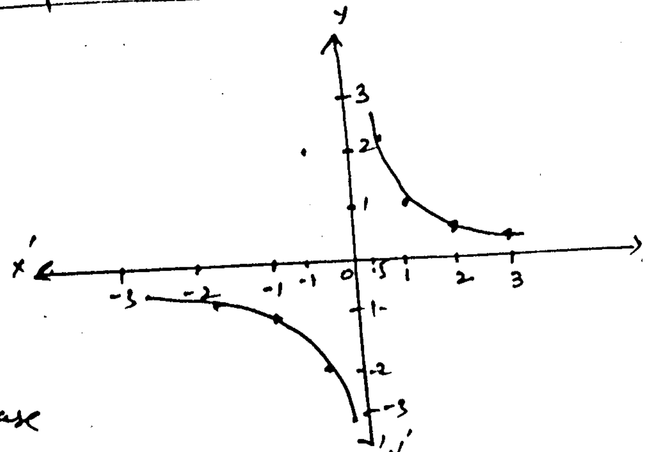
When x is -ve y is also -ve

It's mean the graph of fn: lies Ist and 3rd quadrants.

Also when x increases $f(x)$ decrease

when x decreases $f(x)$ increase.

x	1	-1	-2	2	-3	3	.5	-.5
f(x)	1	-1	-.5	.5	-.33	.33	2	-2



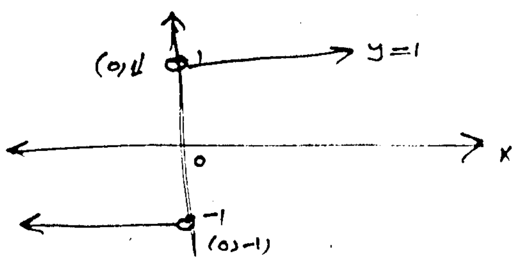
(30) $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$ This is known as Signum (sgn) Function

When $x > 0$ $y = 1$ line ||el to X-axis 1 unit above X-axis

When $x = 0$ $y = 0$ Origin is also part of graph.

When $x < 0$ $y = 1$ line ||el to X-axis 1 unit below X-axis

Note Small Circle at pt (0,1) & (0,-1) are not part of St. line.



(31) **Find the Sup and Inf** (if they exist)

$$\left\{ (-1)^n \left(1 - \frac{1}{n}\right) \quad n = 1, 2, 3 \cdots \right\}$$

Sol: Put values of $n = 1, 2, 3, 4 \cdots$ in given Set, we get

When $n = 1 \quad (-1)^1 \left(1 - \frac{1}{1}\right) = 0$ $\qquad n = 5 \quad (-1)^5 \left(1 - \frac{1}{5}\right) = -\frac{4}{5}$

$n = 2 \quad (-1)^2 \left(1 - \frac{1}{2}\right) = \frac{1}{2}$ $\qquad n = 6 \quad (-1)^6 \left(1 - \frac{1}{6}\right) = 5/6$

$n = 3 \quad (-1)^3 \left(1 - \frac{1}{3}\right) = -\frac{2}{3}$ $\qquad$ - - -

$n = 4 \quad (-1)^4 \left(1 - \frac{1}{4}\right) = \frac{3}{4}$ $\qquad$ - -

$$\left\{ 0, \frac{1}{2}, -2/3, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7} \cdots \right\}$$

Re-arranging, we get

$$\left\{ \cdots -\frac{6}{7}, -\frac{4}{5}, -\frac{2}{3}, 0, \frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \cdots \right\}$$

It is Clear that $\cdots -3, -2, -1$ are Lower bounds of the Set. Since any Real number greater than -1 is not a Lower bound. -1 is the greatest Lower bound.

GLb = Inf = -1

Again $1, 2, 3 \cdots$ are upper bounds of the Set But any Real number Smaller than 1 is not an upper bound. 1 is the Lowest of all.

So LUB or Sup $= 1$

Q33 The Set of all non-negative Integers.

Sol: $S = \{0, 1, 2, 3, \cdots\}$

0 is Lowest of all non-−ve Integer

So GLB or Inf$(S) = 0$

As the Set extends to ∞. So there does not exist LUB or Sup(S)

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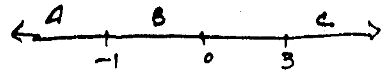
(33) The Set $A = \{ x \in R : 0 < x \leq 3 \}$

Sol $\text{Inf } A = 0 \because 0 \notin A$ and $\text{Sup } A = 3$ But $3 \in A$

Q_{34} The Set $B = \{ x \in R : x^2 - 2x - 3 < 0 \}$

Associated Eq: $x^2 - 2x - 3 = 0$
$(x-3)(x+1) = 0$
$x = -1, 3$



at $x = -1.5$
$(-1.5-3)(-1.5+1)$
$(-4.5)(.5) = +\text{ve}$ False)
at $x=0$ $(-3)(1) = -\text{ve}$ (True)
$x = 4$ $(4-3)(4+1) = +\text{ve}$ False.

$\Rightarrow x^2 - 2x - 3 < 0$
$x^2 - 3x + x - 3 < 0$
$x(x-3) + 1(x-3) < 0$
$(x-3)(x+1) < 0 \longrightarrow (1)$

There are two Cases
(I) $x - 3 > 0$ & $x + 1 < 0$
ii. $x - 3 < 0$ & $x + 1 > 0$

Case-(I) $x > 3$ & $x < -1$ There is no Real number which Satisfied (1) So this is not Possible.

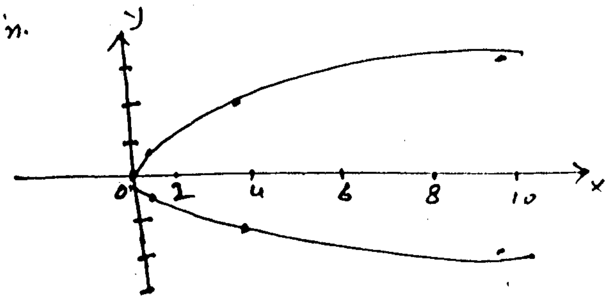
Case-ii $x < 3$ & $x > -1$ Thus $-1 < x < 3$
$\Rightarrow \text{Inf } B = -1$ and $\text{Sup}(B) = 3$

Q_{35} Sketch The graph of given function. Also determine which is $y^2 = x$ the graph of function.

Sol $y^2 = x \longrightarrow (1) \Rightarrow y = \pm\sqrt{x}$

If x is $-$ve y becomes Imaginary So Leave $-$ve value of x
If Put $y = -y$ No Change (1) So it is Symmetric alonge x-axis.
graph of (1) lies +ve Side of x-axis Also $x=0$ & $y=0$
graph passes through Origin.

x	0	1	4	9
$f(x)$	0	± 1	± 2	± 3



$y = \pm\sqrt{x}$ is not a graph of fn: because for one value of x there does not exist Unique Value of y
$\Rightarrow$ Vertical line Cut The graph at two point:-

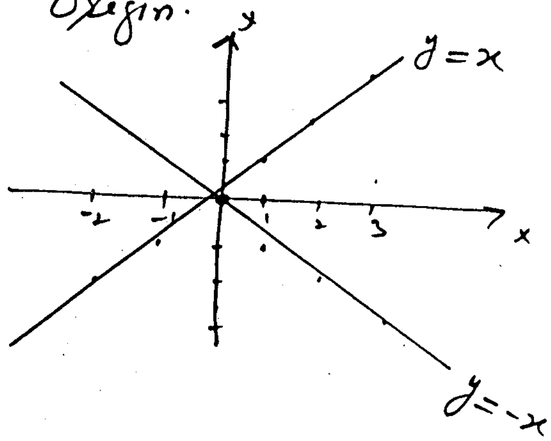
(36) $|x| = |y| \rightarrow (1)$

$x = \pm y$ or $y = \pm x \rightarrow (2)$

$y = x$ & $y = -x$ Pair of St: line Passing Through Origin.

x	0	1	2	3	-1	-2
y=x	0	1	2	3	-1	-2

x	0	1	2	3	-1	-2
y=-x	0	-1	-2	-3	1	2



$y = \pm x$ does not define fn.
for one value of x there does not exist a unique value of y.

---

(37) $x^2 + y^2 = 9 \rightarrow (1)$

$y = \pm\sqrt{9 - x^2}$

When $-3 \le x \le 3$ y will be real.

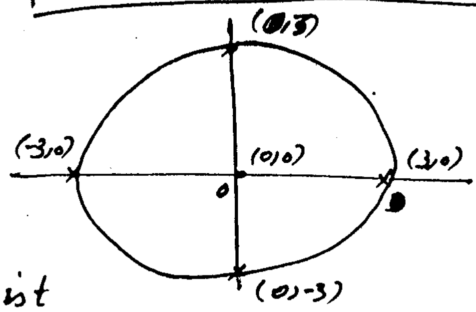
Put $x = -x$ & $y = -y$ No change (1)

It is symmetric both x-axis & y-axis

OR It is sy: at Origin.

(1) is eq: of circle with Centre $(0,0)$ rad $= 3$

X	0	±1	±2	±3
Y	±3	$\pm\sqrt{8}$	$\pm\sqrt{5}$	0



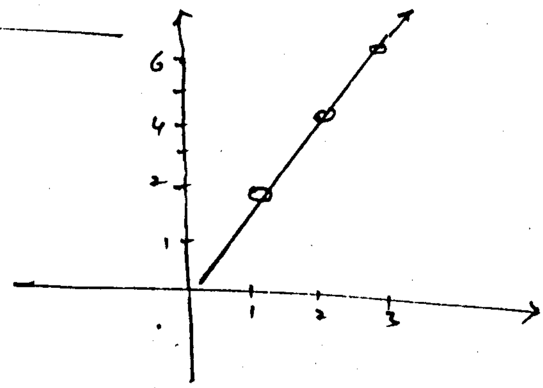
One value of x there does not exist unique value of y (1) is not graph of fn.

---

(38) $y = |x| + x \rightarrow (1)$

Eq (1) Can be written as

$$y = \begin{cases} x + x = 2x & \text{for } x \geq 0 \\ -x + x = 0 & \text{for } x < 0 \end{cases}$$



graph Consists of two St: line
$y = 2x$ when $x \geq 0$
& $y = 0$ when $x < 0$

X	0	1	2	3
Y	0	2	4	6

X	0	1	2	3
y	0	0	0	0

graph is fn.

$\because$ for one value of x there exist unique value of y.

'X'

(39) Find formula for functions $f+g$, fg and $\frac{f}{g}$, where $f(x) = \sqrt{x^2-1}$ and $g(x) = \frac{1}{\sqrt{4-x^2}}$

Also write the domain of each of these functions

Sol:-

i. $(f+g)x = f(x) + g(x)$

$$= \sqrt{x^2-1} + \frac{1}{\sqrt{4-x^2}}$$

ii. $(fg)(x) = f(x) \cdot g(x)$

$$= \sqrt{x^2-1} \cdot \frac{1}{\sqrt{4-x^2}}$$

$$= \sqrt{\frac{x^2-1}{4-x^2}}$$

iii. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$$= \frac{\sqrt{x^2-1}}{1/\sqrt{4-x^2}}$$

$$= \sqrt{x^2-1} \cdot \sqrt{4-x^2}$$

$$= \sqrt{(x^2-1)(4-x^2)}$$

$\because$ $f(x) = \sqrt{x^2-1}$

$f(x)$ will be real when $x^2-1 \geq 0$

$x^2 \geq 1$

$\pm x \geq 1$

$x \geq 1$ & $x \leq -1$

i.e $]-\infty, -1] \cup [1, \infty)$ is domain of $f(x)$

$g(x) = \sqrt{4-x^2}$

$g(x)$ will be real if $4-x^2 \geq 0$

$4 \geq x^2$

$\Rightarrow x^2 \leq 4$

$\pm x \leq 2$

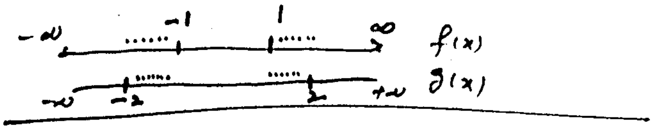
$x \leq 2$ & $x \geq -2$

$[-2, 2]$ is the domain of $g(x)$.

Now domain of each of the functions $f+g$, fg and $\frac{f}{g}$ is

$$\text{Dom} f \cap \text{Dom} g =]-\infty, -1] \cup [1, \infty[\cap [-2, 2]$$

$$= [-2, -1] \cup [1, 2]$$



(40) Find formula for fog and gof, where $f(x) = \sqrt{x^2-3}$ and $g(x) = x^2+3$

Sol:

i. $fog(x) = f(g(x)) = f(x^2+3)$

$$= \sqrt{(x^2+3)^2 - 3}$$

$$= \sqrt{x^4+6x^2+9-3}$$

$$= \sqrt{x^4+6x^2+6}$$

ii. $gof(x) = g[f(x)]$

$$= g(\sqrt{x^2-3})$$

$$= [\sqrt{x^2-3}]^2 + 3$$

$$= x^2 \cancel{-3} \cancel{+3}$$

$$= x^2$$

END

30، رمضان المبارک

13 - 10 - 2007

(5.00 AM)